Quantum Control Theory

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Part 0. From history of control theory/engineering

$$\vdots$$

- Feedback stabilization of steam engines

(J. C. Maxwell "On Governors" 1868)

$$\vdots$$

- Feedback for robust amplifiers (H. S. Black 1927)

$$y = Tu, \quad u = Fy + v \;\; (T \gg 1, F < 1)$$

$$\Rightarrow y = \frac{T}{1 - TF} v \approx -\frac{1}{F} v$$

(T Transfer function/S-matrix for transistor)

$$\vdots$$

Part 1. QSDE and state space equation

§ Classical linear system

Hamiltonian equation

$$\frac{dX(t)}{dt} = \{H(t), X(t)\} = \frac{\partial H}{\partial p}\frac{\partial X}{\partial q} - \frac{\partial H}{\partial p}\frac{\partial X}{\partial q}$$

$X(t) = q(t)$ (position) or $X(t) = p(t)$ (momentum)

$$H(t) = \frac{m\omega^2}{2}q(t)^2 + \frac{1}{2m}p(t)^2 - q(t)u(t)$$

$u(t)$ driving force (input)

⇒ State space equation

$$\underbrace{\frac{d}{dt}\begin{bmatrix} q(t) \\ p(t) \end{bmatrix}}_{\frac{d}{dt}x(t)} = \underbrace{\begin{bmatrix} 0 & \frac{1}{m} \\ -m\omega^2 & 0 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} q(t) \\ p(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C}\underbrace{\begin{bmatrix} q(t) \\ p(t) \end{bmatrix}}_{x(t)} \text{ (output)}$$

open system!
(*i.e.* feedback control based on measurement is possible.)

§ Closed linear quantum system

CCR (Canonical Commutation Relation)

$$[\hat{q}(t), \hat{p}(t)] = \hat{q}(t)\hat{p}(t) - \hat{p}(t)\hat{q}(t) = i\hbar \;\; etc. \quad (i = \sqrt{-1})$$

Heisenberg picture (equation)

$$\frac{d\hat{X}(t)}{dt} = \frac{i}{\hbar}[\hat{H}(t), \hat{X}(t)]$$

$\Rightarrow$

$$\frac{d}{dt}\begin{bmatrix} \hat{q}(t) \\ \hat{p}(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m} \\ -m\omega^2 & 0 \end{bmatrix}\begin{bmatrix} \hat{q}(t) \\ \hat{p}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

(completely the same as the classical one!)

$y(t) =$??? (but no output equation!)

Only open-loop control is possible.

§ **Open linear quantum system**
(quantum system undergoing "continuous measurement")

$\hat{H}(t) = \hat{H} + \hat{H}_c(t)$
$\hat{H}_c(t) = \chi^T N u(t)$ control Hamiltonian term
($u(t)$ Hamiltonian modulation)
($\chi = [\ \cdots \quad \hat{p}_i \quad \hat{q}_i \quad \cdots\ ]^T$)
$\hat{L}(t) = K\chi$ coupling operator

$\hat{H}(t), \hat{L}(t)$ acts on $\mathcal{H}$
$\hat{a}(t), \hat{a}(t)^\dagger$ acts on $\Gamma_s(\mathcal{H})$
(annihilation and creation operators on boson Fock space)
$\hat{U}(t)$ acts on $\mathcal{H} \otimes \Gamma_s(\mathcal{H})$

Belavkin(-Hudson-Parthasarathy) QSDE (1988)
(Shortly Belavkin equation)

$$d\hat{U}(t) = \left[d\hat{a}(t)^{\dagger}\hat{L}(t) - \hat{L}(t)d\hat{a}(t) - \left\{\frac{i}{\hbar}\hat{H}(t) + \frac{1}{2}\hat{L}(t)^{\dagger}\hat{L}(t)\right\}dt\right]\hat{U}(t)$$

Heisenberg picture

$$x(t) = \hat{U}(t)^\dagger \chi \hat{U}(t) = \begin{bmatrix} \vdots \\ \hat{U}(t)^\dagger \hat{p}_i \hat{U}(t) \\ \hat{U}(t)^\dagger \hat{q}_i \hat{U}(t) \\ \vdots \end{bmatrix}$$

By using quantum Itô rule (H.-P. 1984) $d\hat{a}(t)d\hat{a}(t)^\dagger = dt$ *etc.* we obtain

$$\begin{aligned} dx(t) &= Ax(t)dt + Bu(t)dt + B_1 dw(t), \quad x(0) = x_0 \\ dy(t) &= Cx(t)dt + Du(t)dt + D_1 dw(t). \end{aligned}$$

Part 2. Crash course for modern control theory by R. E. Kalman

§ State-space equation

Definition. (Continuous-time automaton)
A *dynamical system* Σ is the collection (A, B, C, D) described by the *state-space equation* given below.

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t) \quad (t \in \mathbb{R}_{\geq 0}) \end{aligned}$$

$u(t) \in \mathbb{R}^m$ (input), $x(t) \in \mathbb{R}^n$ (state),
$y(t) \in \mathbb{R}^p$ (output)
$A \in M_{n\times n}(\mathbb{R})$, $B \in M_{n\times m}(\mathbb{R})$, $C \in M_{p\times n}(\mathbb{R})$, $D \in M_{p\times m}(\mathbb{R})$

§ Controllability

Definition. The pair (A, B) of system Σ is *controllable* if

$$\forall\, x_0, x_1 \in \mathbb{R}^n \text{ and } t_1 \in \mathbb{R}_{>0}$$

$$\exists\, (u(t))_{0 \le t \le t_1} \text{ such that } x(0) = x_0 \text{ and } x(t_1) = x_1.$$

Theorem. *Let*

$$\mathcal{C} = \left[\; B \;\;|\;\; AB \;\;|\;\; \cdots \;\;|\;\; A^{n-1}B \;\right].$$

Then the pair (A, B) is controllable $\Leftrightarrow$ $\operatorname{rank} \mathcal{C} = n$.

Remark. $\mathcal{C}$ is called a *controllability matrix*.

§ Stabilization by state feedback

Theorem. *Suppose that the pair* (A, B) *is controllable. Then* $\exists\, F \in M_{n\times m}(\mathbb{R})$ *such that the system* $\frac{d}{dt}x(t) = (A + BF)x(t)$ *with state feedback* $u(t) = Fx(t)$ *is (asymptotically) stable i.e.* $x(t) \to \vec{0}$ *(*$t \to \infty$*) independently of* $x(0) = x_0$*.*

Remark.

$$\forall\, x(0) = x_0 \in \mathbb{R}^n \;\; x(t) \to \vec{0} \;\; (t \to \infty)$$
$$\Leftrightarrow \sigma(A + BF) \subset \{s \in \mathbb{C} \,|\, \mathrm{Re}(s) < 0\}$$

§ **Observability.**

Definition. (Observability) The pair (A, C) is *observable* if for any $t_1 > 0$ the initial condition $x(0) = x_0$ can be determined from the sequences $(u(t))_{0 \le t \le t_1}$ and $(y(t))_{0 \le t \le t_1}$.

Theorem. *Let*

$$\mathcal{O}^T = \left[\begin{array}{c|c|c|c} C^T & A^T C^T & \cdots & (A^T)^{n-1} C^T \end{array} \right].$$

Then the pair (A, C) *is observable* $\Leftrightarrow$ *rank* $\mathcal{O}^T = n$.

Remark. $\mathcal{O}$ is called an *observability matrix*.

§ **(Luenberger) observer**

Construction of an observer

$$\begin{aligned}\frac{d}{dt}\hat{x}(t) &= A\hat{x}(t) + Bu(t) + K(\underbrace{C\hat{x}(t)}_{\hat{y}(t)} - y(t)) \quad (K \in M_{n\times p}(\mathbb{R})) \\ &= (A + KC)\hat{x}(t) + Bu(t) - Ky(t)\end{aligned}$$

$\Rightarrow \frac{d}{dt}e(t) = (A + KC)e(t) \quad (e(t) = \hat{x}(t) - x(t))$

Theorem. *Suppose that (A, C) is observable. Then $\exists\, K \in M_{n\times p}(\mathbb{R})$ such that $e(t) \to \vec{0}$ for $t \to \infty$.*

§ Stabilization by feedback $u(t) = F\hat{x}(t)$ with estimator $\hat{x}(t)$

Can $u(t) = Fx(t)$ be replaced by $u(t) = F\hat{x}(t)$ without distabilizing the system?
The entire system looks like:

$$\begin{aligned}
\frac{d}{dt}x(t) &= Ax(t) + Bu(t) = Ax(t) + BF\hat{x}(t) \\
y(t) &= Cx(t) \\
\frac{d}{dt}\hat{x}(t) &= A\hat{x}(t) + Bu(t) + K(C\hat{x}(t) - y(t)) \\
u(t) &= F\hat{x}(t).
\end{aligned}$$

Theorem. (Separation Principle) $x(t), e(t) \to \vec{0}$ *for* $t \to \infty$.
(So the answer is YES!)

§ Transfer function

Laplace transform $X(s) = \int_0^\infty x(t)e^{-st}dt$ ($x(0) = x_0 = \vec{0}$) (similarly for $u(t) \mapsto U(s)$ and $y(t) \mapsto Y(s)$)

$$Y(s) = \underbrace{[C(sI - A)^{-1}B + D]}_{T(s)} U(s) \quad \text{(since } \tfrac{d}{dt} \mapsto s\cdot\text{)}$$

$T(s) = C(sI - A)^{-1}B + D$ is called the *transfer function*. We say (A, B, C, D) is a *realization of* $T(s)$.

Remark. $(PAP^{-1}, PB, CP^{-1}, D)$ for $z(t) = Px(t)$ is also a realization of $T(s)$.

§ $T(s)$ as a logic gate

$T(s)$ can be considered as a logic gate.

Example. (NOT)

$$\underbrace{\tau\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\tau\frac{d}{dt}x(t)} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B}\underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}_{u(t)}$$

$$y(t) = \underbrace{\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}}_{C}\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}_{u(t)}$$

$$T(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} 0 & \frac{\tau s-1}{\tau s+1} \\ \frac{\tau s-1}{\tau s+1} & 0 \end{bmatrix}$$

If (A_i, B_i, C_i, D_i) is a realization of $T_i(s)$ $(i = 1, 2)$ then

$$T_2(s)T_1(s) \quad \text{(concatenation)}$$

and

$$T_1(s) \otimes T_2(s) \quad \text{(tensor product)}$$

are also realized using (A_i, B_i, C_i, D_i) $(i = 1, 2)$.

$\S$ **Linear-Quadratic Regulator (LQR)**

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

Cost function $\rightarrow$ minimum (Q, R positive definite matrices)

Minimization problem: $\displaystyle \operatorname*{Min}_{F} \int_0^\infty (x(t)^T Qx(t) + u(t)^T Ru(t))dt$

with feedback $u(t) = Fx(t)$
Solution: $F = -R^{-1}B^T P$
$A^T P + PA - PBR^{-1}B^T P + Q = 0$ (Riccati equation)

§ **Kalman(-Bucy) filter**

$$\begin{aligned} dx(t) &= Ax(t)dt + Bu(t)dt + dw(t), \quad x(0) = x_0 \\ dy(t) &= Cx(t)dt + Du(t)dt + dv(t) \end{aligned}$$

$$\mathbb{E}[w(t)w(t)^T] = Q, \;\; \mathbb{E}[v(t)v(t)^T] = R \quad \text{(Gaussian)}$$

Minimization problem: $\underset{K}{\mathrm{Min}}\, \mathbb{E}[\|\hat{x}(t) - x(t)\|^2]$

with Kalman(-Bucy) filter:

$$d\hat{x}(t) = A\hat{x}(t)dt + Bu(t)dt + K(C\hat{x}(t) - y(t))dt$$

Solution: $K = -PC^T R^{-1}$
$AP + PA^T - PC^T R^{-1} CP + Q = 0$ (Riccati equation)

§ **Separarion principle for LQG=LQR+Kalman(-Bucy) filter**
(LQG=Linear Quadratic Gaussian)

Theorem. (Separation Principle) Replace $u = Fx(t)$ in LQR by $u(t) = F\hat{x}(t)$ ($\hat{x}(t) =$ estimate given by Kalman filter). Then Separation Principle holds for LQR+Kalman filter.

§ H^∞**-control (or optimization), robust control** (G. Zames)

Simple Example to give a flavor of H^∞-control problem

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) + w(t) \\ u(t) &= Fx(t) \end{aligned}$$

$\Rightarrow X(s) = [sI - (A + BF)]^{-1} W(s)$

Replace the Gaussian assumption $w(t) \sim N(\bar{w}, Q)$ by the assumption $w(t) \in L^2(0, \infty)$.

Minimization problem: $\operatorname*{Min}_F \|[sI - (A + BF)]^{-1}\|_{H^\infty}$

Part 3. Scattering theory and systems

§ Scattering for quantum two-body problem

$$\hat{H} = -\Delta + V(x) = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + V(x_1, \ldots, x_n)$$

Schrödinger equations:

$$\frac{\partial}{\partial t}|\psi(t)\rangle = -\frac{i}{\hbar}\hat{H}|\psi(t)\rangle, \quad |\psi(0)\rangle = |f\rangle$$

$$\Rightarrow |\psi(t)\rangle = e^{-\frac{i}{\hbar}t\hat{H}}|f\rangle = \hat{U}(t)|f\rangle \text{ perturbed solution}$$

The stationary Schrödinger equation

$$(\hat{H} - k^2)|\psi\rangle = 0, \quad k > 0$$

$+$ incoming and outgoing Sommerfeld radiation condition
(a boundary condition at infinity)

The scattering matrix (or S-matrix) $\mathcal{S}\colon L^2(\mathbb{R}_+, \mathcal{N}) \to L^2(\mathbb{R}_+, \mathcal{N})$
is defined as a mapping $\tilde{f}_{\text{in}} \mapsto \tilde{f}_{\text{out}}$:

$$\tilde{f}_{\text{out}}(k\vec{\omega}) = \mathcal{S}(k)\tilde{f}_{\text{in}}(k\vec{\omega}) \quad (\vec{\omega} \text{ unit sphere in } \mathbb{R}^n).$$

Remark. Heisenberg's scattering (or S-)matrix is

$$|f(+\infty)\rangle = S|f(-\infty)\rangle.$$

§ **Scattering model** $(\{U_m(t)\}_{t\in\mathbb{R}}, \mathcal{M})$
(Continuous-time Turing machine)

$$U_m(t) \text{ acts on } \mathcal{M} = \underbrace{L^2(\mathbb{R}_-, U)}_{\text{Incoming space}} \oplus \underbrace{X}_{\text{Scattering space}} \oplus \underbrace{L^2(\mathbb{R}_+, Y)}_{\text{Outgoing space}}$$

Write an element of the Hilbert space $\mathcal{M}$ as

$$h = (u_-, x_0, y_+) \in \mathcal{M},$$

where

$$u_- = u_-(\tau) \in L^2(\mathbb{R}_-, U) \ \ (\tau \in \mathbb{R}_-), \quad x_0 \in X,$$

$$y_+ = y_+(\tau) \in L^2(\mathbb{R}_+, Y) \ \ (\tau \in \mathbb{R}_+).$$

The (infinitesimal) generator L of $\{U_m(t)\}_{t\in\mathbb{R}}$ defined by

$$Lh = \lim_{t\to 0_+} \frac{1}{t}(U_m(t)h - h), \quad h \in \mathrm{Dom}(L)$$

is given by $Lh = (u_1, x_1, y_1)$, where

$$\begin{aligned} u_1(\tau) &= -(\tfrac{d}{d\tau}u_-)(\tau) \qquad (\tau \in \mathbb{R}_-), \\ x_1 &= Ax_0 + Bu_-(0), \\ y_1(\tau) &= -(\tfrac{d}{d\tau}y_+)(\tau) \qquad (\tau \in \mathbb{R}_+), \end{aligned}$$

with boundary condition

$$y_+(0) = Cx_0 + Du_-(0).$$

Transfer function and scattering matrix are related as follows.

$$T(s) = C(sI - A)^{-1}B + D = \mathcal{S}(k) \ \ (s = -ik^2)$$

§ **Single Electron Transistor (SET) as a scattering system**

$$\mathcal{M} = \underbrace{L^2(\mathbb{R}_-, U)}_{\text{Source}} \oplus \underbrace{X}_{\substack{\text{Island}\\ \text{(Quantum dot)}}} \oplus \underbrace{L^2(\mathbb{R}_+, Y)}_{\text{Drain}}$$

$$T(s) = C(sI - A)^{-1}B + D(= \mathcal{S}(k))$$

Feedback is given as a boundary condition

$$u_-(0) = Fx_0.$$